MINIMAL SURFACES IN $\mathbb{S}^3$ AND YAU'S CONJECTURE

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Abstract. We list some known facts and open problems about minimal surfaces in $\mathbb{S}^3$. And we sketch a proof of Yau's conjecture for Lawson's minimal surfaces and Karcher-Pinkall-Sterling's minimal surfaces.

1. Minimal Surfaces in $\mathbb{S}^3$

The catenoid, the helicoid, Scherk's surfaces, and some triply periodic minimal surfaces had been the only complete embedded minimal surfaces known to exist in $\mathbb{R}^3$ until Costa and Hoffman-Meeks constructed minimal surfaces of arbitrary genus in 1980's. In the three-dimensional sphere $\mathbb{S}^3$ Lawson [L1] constructed compact embedded minimal surfaces of arbitrary genus, and Karcher-Pinkall-Sterling [KPS] added some more examples. Both in $\mathbb{R}^3$ and in $\mathbb{S}^3$, a paucity of examples has been a main obstacle to the study of embedded minimal surfaces. Still, we know some a priori properties of compact minimal surfaces in $\mathbb{S}^3$ as follows.

(1) An immersed minimal sphere in $\mathbb{S}^3$ is totally geodesic. (Almgren)

(2) The center of gravity of a compact minimal submanifold of $\mathbb{S}^n$ is at the origin.

(3) Two minimal hypersurfaces of $\mathbb{S}^n$ must intersect each other. (Frankel [F])

(4) For each integer g there is a compact embedded minimal surface of genus g in $\mathbb{S}^3$. (Lawson [L1])

(5) In $\mathbb{S}^3$ there exist compact embedded minimal surfaces of genus 3, 5, 6, 7, 11, 17, 19, 73, and 601. (Karcher-Pinkall-Sterling [KPS])

(6) To each complete minimal surface in $\mathbb{S}^3$ there is a complete locally isometric surface of constant mean curvature in $\mathbb{R}^3$. (Lawson [L1])

(7) Embedded minimal surfaces in $\mathbb{S}^3$ cannot have knotted handles. (Lawson [L2])

(8) If a compact branched minimal surface and a great circle in $\mathbb{S}^3$ are disjoint, then they are linked. (Solomon [S])

(9) The space of compact embedded minimal surfaces in $\mathbb{S}^3$ is compact in C^k topology. (Choi-Schoen [ChS])

(10) The Morse index of compact minimal surfaces in $\mathbb{S}^3$ is 1 for the great sphere, 5 for the Clifford torus $\mathbb{S}^1(1/\sqrt{2}) \times \mathbb{S}^1(1/\sqrt{2})$, and higher for the others. (Urbano [U])

(11) If the boundary of a compact immersed orientable and stable minimal hypersurface Σ in $\mathbb{S}^n$ lies in a great sphere $\mathbb{S}^{n-1}$, then $\Sigma \subset \mathbb{S}^{n-1}$. (Ros [R])
(12) If the boundary of a compact immersed orientable minimal hypersurface Σ in $\mathbb{S}^n$ lies in a great sphere $\mathbb{S}^{n-1}$, then $\mathrm{Vol}(\Sigma) \geq \frac{1}{2}\mathrm{Vol}(\mathbb{S}^{n-1})$, with equality only if Σ is a hemisphere. (Ros [R])
(13) The only compact embedded orientable minimal surface in $\mathbb{S}^3$ that bounds a great circle is the hemisphere. (Hardt-Simon [HS])
(14) If a compact embedded orientable minimal surface Σ in $\mathbb{S}^3$ bounds two orthogonally intersecting great circles, then Σ is a half of the Clifford torus. (Hardt-Rosenberg [HR])
(15) In $\mathbb{S}^n$ any great sphere divides a compact embedded minimal hypersurface into two connected pieces. (Ros [R])
(16) The Gauss map of a minimal surface $\Sigma \subset \mathbb{S}^3$ gives a branched minimal surface Σ^* in $\mathbb{S}^3$. Moreover, $\Sigma^{**} = \Sigma$. (Lawson [L1])
(17) If Σ is a compact embedded minimal torus in $\mathbb{S}^3$, then its Gauss image Σ^* is also embedded. (Ros [R])
(18) For each conformal structure on a compact surface, there exists at most one metric admitting a minimal immersion into $\mathbb{S}^n$ on which the first eigenvalue of the Laplacian equals two. (Montiel-Ros [MR])
(19) The only minimal torus in $\mathbb{S}^3$ on which the first eigenvalue of the Laplacian equals two is the Clifford torus. (Montiel-Ros [MR])

Now let's consider some open problems and conjectures for minimal surfaces in $\mathbb{S}^n$:

(1) Is there a complete immersed minimal surface in $\mathbb{S}^3$ which is disjoint from a great sphere $\mathbb{S}^2$? This is an $\mathbb{S}^3$-version of Calabi's question which was solved affirmatively by Nadirashvili [N].
(2) For any given integer g there are only finitely many noncongruent minimal surfaces of genus g in $\mathbb{S}^3$.
(3) (Lawson's conjecture) The only embedded minimal torus in $\mathbb{S}^3$ is the Clifford torus. Combining with (2), one may even conjecture that the only compact embedded minimal surfaces are the surfaces $\xi_{m,k}$ constructed by Lawson in [L1].
(4) (Yau's conjecture [Y]) The first eigenvalue of the Laplacian on a compact embedded minimal hypersurface Σ^n in $\mathbb{S}^{n+1}$ is equal to n.

Let $x_1, ..., x_m$ be the rectangular coordinates of $\mathbb{R}^m$ and let $X := (x_1, ..., x_m)$. Given a submanifold M of $\mathbb{R}^n$, it is well known that

$$\Delta_M X = \vec{H},$$

where $\vec{H}$ is the mean curvature vector of M. Therefore $x_1, ..., x_m$ are harmonic functions on a minimal submanifold $\Sigma \subset \mathbb{R}^m$. If Σ^n is minimal in $\mathbb{S}^{m-1}$, then the cone $O \times\!\!\!\!\times \Sigma$ is also minimal in $\mathbb{R}^m$. Therefore $\Delta_\Sigma X$ must be perpendicular to $\mathbb{S}^{m-1}$ and hence $\Delta_\Sigma X$ is parallel to X. Then it is not difficult to show that

$$\Delta_\Sigma X + nX = 0.$$

Therefore $x_1, ..., x_m$ are eigenfunctions of Δ with eigenvalue n on the n-dimensional minimal submanifold Σ of $\mathbb{S}^{m-1}$.

Thus it was natural for Yau to propose his conjecture as above. Yau's conjecture does not concern minimal surfaces with nonempty self intersection and minimal surfaces of high codimension because a minimal surface of revolution of large area in $\mathbb{S}^3$ and the Veronese surface in $\mathbb{S}^4$ have the first eigenvalue much smaller than two.

It may have been just out of curiosity that Yau made his conjecture. But Montiel-Ros [MR] showed that Yau's conjecture has a geometric implication: If Yau's conjecture is true, then the Clifford torus is the only embedded minimal torus in $\mathbb{S}^3$, i.e., Lawson's conjecture is true as well. It should be mentioned that Choi-Wang [CW] proved that the first eigenvalue on Σ^n is at least $n/2$.

There is a well-known theorem by Courant that the first eigenfunction of Δ on Σ has *two* nodal domains. In this regard it is very interesting to note that a compact embedded minimal surface in $\mathbb{S}^3$ has *two-piece property*: Ros [R] proved that any great sphere in $\mathbb{S}^3$ divides a compact embedded minimal surface Σ into two connected pieces. However, if Yau's conjecture is true, then Ros's two-piece property follows from Courant's theorem. Indeed, if 2 is the first eigenvalue of Δ, then Courant's nodal theorem for the linear function $\phi = a_1x_1 + ... + a_4x_4$ with $\phi|_{\mathbb{S}^2} = 0$ implies the two-piece property.

Therefore, now that the two-piece property holds, one might presume that Yau's conjecture should be true. As a matter of fact, the author and M. Soret [CS] found that by using Courant's nodal theorem and Ros's two piece property one can prove Yau's conjecture for minimal surfaces in $\mathbb{S}^3$ which are sufficiently symmetric (as much symmetric as Lawson's surfaces and Karcher-Pinkall-Sterling's surfaces).

2. Yau's Conjecture

In this section we briefly outline the arguments of our paper [CS].

Lemma 1. *If the boundary of a compact immersed orientable and stable minimal hypersurface Σ^n in $\mathbb{S}^{n+1}$ lies in a great sphere, then Σ is totally geodesic.*

Proof. See Lemma 1 of [CS]. □

Theorem 1. *Any great sphere in $\mathbb{S}^{n+1}$ divides a compact embedded minimal hypersurface Σ of $\mathbb{S}^{n+1}$ into two connected pieces.*

Proof. See Theorem 1 of [CS]. □

Lemma 2. *Let G be a group of reflections in $\mathbb{S}^3$. Assume that a minimal surface $\Sigma \subset \mathbb{S}^3$ is invariant under G. If the first eigenvalue of Δ on Σ is less than 2, then the first eigenfunction must be invariant under G.*

Proof. (*Sketch*) Let $\sigma \in G$ be the reflection across a great sphere Π in $\mathbb{S}^3$ and let ϕ be an eigenfunction on Σ corresponding to the first eigenvalue λ_1. Note that $\phi \circ \sigma$ is again an eigenfunction with eigenvalue λ_1. Consider

$$\psi(x) := \phi(x) - \phi \circ \sigma(x).$$

If ψ is the null function then ϕ is invariant under σ. If $\psi \not\equiv 0$ then ψ itself is an eigenfunction with eigenvalue λ_1. Furthermore its nodal set contains $\Sigma \cap \Pi$. But Courant's nodal theorem implies that ψ vanishes only on $\Sigma \cap \Pi$. Let D_1, D_2 be the components of $\Sigma \setminus \Pi$ such that ψ is positive on D_1 and negative on D_2. By Ros's two-piece property D_1, D_2 are each connected. One can find a linear function of $\mathbb{R}^4$ $\xi = a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4$ that vanishes on Π and is positive on D_1. Clearly ξ is orthogonal to ψ on Σ. But ψ and ξ have the same sign on $D_1 \cup D_2$, which contradicts the orthogonality of ψ and ξ. Therefore ψ must vanish on Σ. This completes the proof as σ is an arbitrary element of G. □

Theorem 2. *Let Σ be a minimal surface in $\mathbb{S}^3$ which is invariant under a group G of reflections. Suppose that the fundamental domain of G in $\mathbb{S}^3$ is a tetrahedron T. If the fundamental patch $S = \Sigma \cap T$ is simply connected and has four edges, then the first eigenvalue of the Laplacian on Σ equals 2.*

Proof. Suppose $\lambda_1 < 2$. Let ϕ be an eigenfunction with eigenvalue λ_1 on Σ and $N \subset \Sigma$ the nodal set of ϕ. From Lemma 2 it follows that $S \setminus N$ has at least two connected components. Since S is simply connected one can find a face F of T and a component D of $S \setminus N$ such that ∂D is disjoint from F. Let Π be the great sphere containing F and let $\hat{D}$ be the mirror image of D across Π. Denote by D_1, D_2, D_3 the components of $\Sigma \setminus N$ containing $D, \hat{D}$ and intersecting Π, respectively. We claim that D_1, D_2, D_3 are all distinct. D_2 is the mirror image of D_1 and D_3 is nonempty and symmetric with respect to Π. See [CS] for the details. Therefore ϕ has at least three nodal domains, which contradicts Courant's nodal theorem. Thus $\lambda_1 = 2$. □

Lemma 3. *Lawson's minimal surfaces $\xi_{m,k}$ can also be constructed in the same way as Karcher-Pinkall-Sterling's surfaces are constructed.*

Proof. See Section 2 of [CS]. □

Corollary 1. *The first eigenvalue of the Laplacian on Lawson's embedded minimal surfaces $\xi_{m,k}$ and Karcher-Pinkall-Sterling's minimal surfaces in $\mathbb{S}^3$ is equal to 2.*

Theorem 3. *Let Σ be a compact embedded minimal surface in S^3 which is invariant under a group or reflections, and let $D \subset \Sigma$ be a fundamental patch in a tetrahedron* *of the tessellation. If D is simply connected and has at most five edges, then* $\lambda_1(\Sigma) = 2$.

Proof. See Theorem 3 of [CS]. □

Remark. If the fundamental patch D has six edges, λ_1 may still equal two in case the genus of the minimal surface is sufficiently small. See Section 6 of [CS] for the details.

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